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Symmetries and phase structure of the layered sine-Gordon model

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Abstract

The phase structure of the layered sine-Gordon (LSG) model is investigated in terms of symmetry considerations by means of a differential renormalization group (RG) method, within the local potential approximation. The RG analysis of the general N -layer model provides us with the possibility of considering the dependence of the vortex dynamics on the number of layers. The Lagrangians are distinguished according to the number of zero eigenvalues of their mass matrices. The number of layers is found to be decisive with respect to the phase structure of the N -layer models, with neighbouring layers being coupled by terms quadratic in the field variables. It is shown that the LSG model with N layers undergoes a Kosterlitz–Thouless-type phase transition at the critical value of the parameter $\beta_c^2 = 8N\pi$. In the limit of infinitely many layers the LSG model can be considered as the discretized version of the three-dimensional sine-Gordon model which has been shown to have a single phase within the local potential approximation. The infinite critical value of the parameter β_c^2 for the LSG model in the continuum limit ($N \rightarrow \infty$) is consistent with the latter observation.

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1. Introduction

The renormalization of sine-Gordon (SG) type models represents a challenge in quantum field theory, where the usual strategies are based on the Taylor expansion of the interaction Lagrangian. However, in the case of an SG-type scalar field theory where the self-interaction is given by a periodic term (periodic in the field variable), any truncation of the Taylor expansion of the potential violates the essential symmetry of the model. As is well known, the phase structure of the system crucially depends on the symmetries of the interaction Lagrangian in the field variable. Therefore, in order to obtain the low-energy effective theory and to map out

the phase structure of an SG-type model, one has to use a method which retains the periodicity of the system.

The phase structure of the ‘pure’ SG model which is periodic in the internal space spanned by the field variable has been investigated in great detail [1–3], and as it is well known, the model has two phases separated by the critical value of the parameter, $\beta_c^2 = 8\pi$ [4, 5]. Another interesting subject concerns the ‘massive’ SG model [6, 7] where the periodicity is broken by the explicit mass term of the Lagrangian. The massive SG model has a single phase; all the coupling constants of the model are relevant parameters independently of β . It is an interesting subject to consider an SG-type theory which combines the ‘features’ of the massless and massive SG models where the periodicity is broken only partially. The following generalization of the SG model which is called the layered sine-Gordon (LSG) model [7, 8] belongs to the latter category,

$$\mathcal{L}_{\text{LSG}} = \frac{1}{2} \sum_{i=1}^N (\partial\varphi_i)^2 + \frac{1}{2} J \sum_{i=1}^{N-1} (\varphi_{i+1} - \varphi_i)^2 + U(\varphi_1, \dots, \varphi_N), \quad (1)$$

where each φ_i is a one-component Lorentz scalar field and the second term corresponds to the coupling between the SG models (i.e. layers). $U(\varphi_1, \dots, \varphi_N)$ is assumed to be periodic but the periodicity is broken (partially) by the interlayer coupling terms. The LSG model has relevance in high-energy and low-temperature physics. The sine-Gordon model with N layers can be considered as the bosonized version of the N -flavour Schwinger model [7]. Another suitable generalization of the SG model is the $SU(N)$ Thirring model [9, 10]. The LSG model with $N = 2$ layers has been used to describe the vortex dominated properties of high- T_c superconductors which have a layered structure [11, 12].

Recently, the LSG model with two coupled layers has been analysed in the framework of the non-perturbative Wegner–Houghton (WH) renormalization group (RG) method which retains the periodicity of the model [13]. The WH–RG approach [14] is incompatible with the derivative expansion due to the sharp momentum cutoff used. However, it represents one of the most straightforward implementations of a functional RG method. As such, it is a rather powerful tool for the analysis of the RG flow of theories with periodic self-interactions, including situations with more than one interacting field and the higher harmonics which may be generated during the RG evolution for periodic self-interactions. The WH–RG method provides us with a suitable tool for the investigation of the phase structure of these periodic field theories. In this paper, we would like to present a further contribution to the study of related models by means of the WH–RG method performed for the LSG model with N layers.

We present an explicit rotation in the internal space of the field variables, which allows us to decompose the Lagrangians into ‘periodic’ and ‘non-periodic’ fields. In this paper we refer to a field variable whose self-interaction is characterized by a periodic function with and without an explicit symmetry breaking mass term as a ‘non-periodic’ and ‘periodic’ mode, respectively.

The purpose of this rotation is twofold. First, we would like to compare the result of our RG analysis to that of the perturbative treatment discussed in [15] where the rotated N -layer SG model is studied. In the infrared (IR) region, with $k \ll M$ (with M being the mass eigenvalue and k is the momentum cutoff), it is allowed to use perturbation theory but only for the non-periodic modes [16]. Second, by using the rotation we would like to demonstrate that the internal symmetry in the field variable is decisive for the phase structure of the LSG model.

The non-periodic modes have a trivial tree-level scaling, which is consistent with the explicit breaking of the internal periodicity in the field variable. In the limit of a vanishing interlayer coupling J , the coupled N -layer model approaches the sum of N decoupled

sine-Gordon models, each of which has a Kosterlitz–Thouless-type phase transition at the critical value $\beta_c^2 = 8\pi$ (see [1–3, 5]). For an N -layer model with a non-vanishing coupling J , we show here that there exists exactly one periodic mode which has two types of scaling behaviour separated by the critical value $\beta_c^2 = 8N\pi$ where N is the number of layers.

Our paper is organized as follows. In section 2, we define the layered SG model and discuss the connection to the two-dimensional (2D) and three-dimensional (3D) SG models. We then give basic relations used for the Wegner–Houghton RG method [14] and derive the mass-corrected ultraviolet (UV) WH–RG equation in section 3. In section 4, the flavour-doublet and in section 5, the flavour-triplet LSG models are analysed by means of the UV mass-corrected WH–RG method in detail. The generalization to N layers is also studied. In section 6, the UV-RG evolution of the 3D-SG model is investigated and compared to that of the LSG model with N layers. Finally, we conclude with a summary in section 7.

2. Layered sine-Gordon model

The LSG model belongs to a wider class of massive SG-type theories due to the interlayer coupling which can be considered as a mass term. In general, the bare Lagrangian for the massive SG model with N layers is [13, 15]

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^N (\partial\varphi_i)^2 + \frac{1}{2} \sum_{i,j} \varphi_i M_{i,j}^2 \varphi_j + U(\varphi_1, \dots, \varphi_N), \quad (2)$$

where φ_i is a one-component scalar field, the theory is constructed in $d = 2$ dimensions in Euclidean metric and the periodic self-interaction is given by the term

$$U(\varphi_1, \dots, \varphi_N) = U\left(\varphi_1 + \frac{2\pi}{\beta_1}, \dots, \varphi_N + \frac{2\pi}{\beta_N}\right). \quad (3)$$

The model has a global $Z(2)$ discrete symmetry $\varphi_i \rightarrow -\varphi_i$. By applying an orthogonal transformation on the flavour multiplet $(\varphi_1, \varphi_2, \dots, \varphi_N)$, the massive SG model transforms into a similar one with transformed period lengths in the internal space. Since the global $O(N)$ rotation does not mix the field fluctuations with different momenta, the scaling laws and the phase structure should be the same for all the rotated models.

The mass matrix M_{ij}^2 ($i, j = 1, 2, \dots, N$) is symmetric and positive semidefinite and assumed to have a special ‘interlayer’ structure,

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^N (\partial\varphi_i)^2 + \frac{1}{2} \sum_{i=1}^N M_i^2 \varphi_i^2 + \frac{1}{2} J \sum_{i=1}^{N-1} (\varphi_{i+1} - \varphi_i)^2 + U(\varphi_1, \dots, \varphi_N), \quad (4)$$

where the explicit mass terms are M_i^2 ($i = 1, \dots, N$) and J describes the interaction between the layers. Since the layers are assumed to be equivalent $M_i^2 \equiv M^2$ for $i = 1, \dots, N$ is a natural choice. The symmetries and phase structure of the massive SG model (4) with $N = 2$ layers have already been discussed in [13]. It was demonstrated that the number of zero eigenvalues of the mass matrix is found to be decisive with respect to the phase structure of the model. The mass eigenvalues of the layered system (4) for $N = 2, 3, 4$ layers are $[M^2, 2J + M^2]$, $[M^2, J + M^2, 3J + M^2]$ and $[M^2, 2J + M^2, (2 + \sqrt{2})J + M^2, (2 - \sqrt{2})J + M^2]$, respectively. Consequently, for vanishing explicit masses ($M^2 = 0$) the layered model has always a single zero mass eigenvalue and as it was shown for the ($N = 2$)-layer case, it undergoes a phase transition. This can also be understood in terms of symmetry considerations. In the presence (absence) of explicit mass terms the periodic symmetry of the layered model (4) is broken entirely (partially). In this paper we would like to clarify this general statement

by considering the phase structure of the N -layer SG model by means of the differential RG approach.

For vanishing explicit mass terms ($M^2 = 0$) the Lagrangian (4) takes the form of equation (1),

$$\mathcal{L}_{\text{NL SG}} = \frac{1}{2} \sum_{i=1}^N (\partial \varphi_i)^2 + \frac{1}{2} J \sum_{i=1}^{N-1} (\varphi_{i+1} - \varphi_i)^2 + U(\varphi_1, \dots, \varphi_N), \quad (5)$$

with $\beta_i = \beta$ (for $i = 1, 2, \dots, N$). The LSG model with $N = 2$ layers has been proposed as an adequate description of the vortex dominated properties of strongly anisotropic high transition temperature superconductors which have a layered structure. In this case the periodic term has a simple structure

$$\mathcal{L}_{2\text{LSG}} = \frac{1}{2} \sum_{i=1}^2 (\partial \varphi_i)^2 + \frac{1}{2} J (\varphi_2 - \varphi_1)^2 + u [\cos(\beta \varphi_1) + \cos(\beta \varphi_2)], \quad (6)$$

where u corresponds to the fugacity parameter of the vortex system, β is related to the temperature and the second term describes the weak Josephson coupling between the superconducting layers [11].

Finally, we would like to demonstrate that in the limit $N \rightarrow \infty$ the LSG model can be considered as the discretized version of the 3D-SG model. The 3D-SG model has the following action:

$$S = \int d^3 r \left[\frac{1}{2} (\partial_\mu \varphi_{3\text{D}})^2 + u_{3\text{D}} \cos(\beta_{3\text{D}} \varphi_{3\text{D}}) \right], \quad (7)$$

where $\varphi_{3\text{D}} \equiv \varphi_{3\text{D}}(x, y, z)$ is a one-component scalar field and $\beta_{3\text{D}}, u_{3\text{D}}$ are the dimensionful parameters of the theory. The model is constructed in $d = 3$ spatial dimensions with an Euclidean metric. The anisotropic 3D-SG model reads

$$S = \int d^3 r \left[\frac{1}{2\beta_{\parallel}^2} [(\partial_x \theta)^2 + (\partial_y \theta)^2] + \frac{1}{2\beta_{\perp}^2} (\partial_z \theta)^2 + u_{3\text{D}} \cos(\theta) \right], \quad (8)$$

where $\theta = \varphi_{3\text{D}} \beta_{3\text{D}}$ is introduced. In the isotropic limit $\beta_{\parallel} = \beta_{\perp} \equiv \beta_{3\text{D}}$ is assumed. Rescaling the field $\Phi = \theta/\beta_{\parallel}$, the action (8) becomes

$$S = \int d^3 r \left[\frac{1}{2} [(\partial_x \Phi)^2 + (\partial_y \Phi)^2] + \frac{\beta_{\parallel}^2}{2\beta_{\perp}^2} (\partial_z \Phi)^2 + u_{3\text{D}} \cos(\beta_{\parallel} \Phi) \right]. \quad (9)$$

In the case of very strong anisotropy, the continuous derivation and the integration in the z -direction is replaced by finite difference and summation, respectively,

$$\partial_z \Phi(x, y, z) \rightarrow \frac{\Phi(x, y, z+s) - \Phi(x, y, z)}{s}, \quad \int dz \rightarrow \sum_{z=1}^N s, \quad (10)$$

where s is the interlayer distance. Using this discretization, one arrives at the LSG model with N layers,

$$S = \int d^2 r \left[\frac{1}{2} \sum_{i=1}^N (\partial \varphi_i)^2 + \frac{1}{2} J \sum_{i=1}^{N-1} (\varphi_{i+1} - \varphi_i)^2 + u \sum_{i=1}^N \cos(\beta \varphi_i) \right], \quad (11)$$

where $\varphi_i(x, y) \equiv \sqrt{s} \Phi(x, y, z = i)$, $J \equiv \beta_{\parallel}^2 / (\beta_{\perp}^2 s^2)$, $\beta \equiv \beta_{\parallel} / \sqrt{s}$ and $u \equiv s u_{3\text{D}}$ are introduced. Therefore, in the continuum limit $N \rightarrow \infty$ the LSG model can be considered as the discretized version of the 3D-SG model and for $N = 1$ the LSG model reduces to the 2D-SG model.

3. Wegner–Houghton renormalization group method

In order to map out the phase structure of the layered system, we perform an RG analysis for the layered SG model by means of the differential RG approach in momentum space where the blocking transformations are realized by successive elimination of the field fluctuations according to their decreasing momentum in infinitesimal steps [17]. The high-frequency modes are integrated out above the moving momentum cutoff k and the physical effects of the eliminated modes are encoded in the scale dependence of the coupling constants. The elimination of the modes above the moving scale k is complete in Wegner's and Houghton's method (WH–RG) [14] because of the sharp momentum cutoff. The WH method provides a functional RG equation for the blocked action. In order to solve the WH–RG equation, one has to project it to a particular functional subspace. Therefore, one generally assumes that the blocked action contains only local interactions; then let us expand it in powers of the gradient of the field and truncate this expansion at a given order, for technical reasons [18]. Here we restrict ourselves to the leading order of the gradient expansion, i.e. to the local-potential approximation (LPA). The blocked action for the LSG model with N layers reads

$$S_k = \int d^2x \left[\frac{1}{2} \sum_{i=1}^N (\partial \varphi_i)^2 + V_k(\varphi_1, \dots, \varphi_N) \right], \quad (12)$$

where k is the running momentum cutoff and $V_k(\varphi_1, \dots, \varphi_N)$ is the blocked potential which has the following form:

$$V_k(\varphi_1, \dots, \varphi_N) = \frac{1}{2} \underline{\varphi}^T \underline{\underline{M}}_k^2 \underline{\varphi} + U_k(\varphi_1, \dots, \varphi_N), \quad (13)$$

where U_k is the periodic part of the blocked potential and $\underline{\underline{M}}_k^2$ represents the scale-dependent mass matrix. Note that the momentum scale dependence is encoded in the coupling constants of the model. The WH–RG equation in LPA has been derived for two interacting scalar fields in [13, 19]. The generalization for the N -layer SG model is straightforward and can be written as

$$k \partial_k V_k(\varphi_1, \dots, \varphi_N) = -\frac{k^2}{4\pi} \ln \left(\frac{\det [\delta_{ij} k^2 + V_k^{ij}]}{k^{2N}} \right), \quad (14)$$

where δ_{ij} is the Kronecker delta, $V_k^{ij} = \partial_{\varphi_i} \partial_{\varphi_j} V_k(\varphi_1, \dots, \varphi_N)$ is the second derivative of the dimensionful blocked potential with respect to the field variables. We then introduce dimensionless parameters in order to remove the trivial scale dependence of the coupling constants. The WH–RG equation in LPA for the dimensionless blocked potential reads

$$(2 + k \partial_k) \tilde{V}_k(\varphi_1, \dots, \varphi_N) = -\frac{1}{4\pi} \ln (\det [\delta_{ij} + \tilde{V}_k^{ij}]), \quad (15)$$

where $\tilde{V}_k = k^{-2} V_k$ is introduced. All dimensionless quantities will be denoted with a tilde superscript in the following. We recall that in $d = 2$ dimensions the scalar fields carry no physical dimension, so that $\varphi_i = \tilde{\varphi}_i$ and hence $\beta = \tilde{\beta}$.

Inserting the dimensionless form of the ansatz (13) into the WH–RG equation (15), the right-hand side turns out to be periodic, while the left-hand side contains both periodic and non-periodic parts. The non-periodic part contains the mass term and we obtain the trivial tree-level evolution for the dimensionless mass parameters $\tilde{M}_{ij}^2(k)$,

$$\tilde{M}_{ij}^2(k) = \tilde{M}_{ij}^2(\Lambda) \left(\frac{k}{\Lambda} \right)^{-2}, \quad (16)$$

where $\tilde{M}_{ij}^2(\Lambda)$ is the initial value for the mass term at the UV momentum cutoff Λ . Therefore, the dimensionful mass terms have no evolution, i.e. M_{ij}^2 are scale independent. The RG flow equation

$$(2 + k\partial_k)\tilde{U}_k(\varphi_1, \dots, \varphi_N) = -\frac{1}{4\pi} \ln(\det[\delta_{ij} + \tilde{V}_k^{ij}]) \quad (17)$$

stands for the dimensionless periodic piece of the blocked potential.

In order to obtain the scale dependence of the coupling constants, one has to solve the differential equation (17) which can be done only numerically. However, analytic solutions are available by considering asymptotic approximations of equation (17). In the case of the UV approximation, the potential is assumed to be much smaller than the momentum cutoff k and the logarithm can be linearized in equation (17). In order to obtain reliable UV scaling laws which can be used to determine the phase structure of the layered model, one has to incorporate the effect of the mass terms (coupling between the layers). Therefore, one should use the ‘mass-corrected’ UV approximation of equation (17) which has been discussed in [13]. We briefly summarize here the derivation of the mass-corrected UV-RG where the argument of the logarithm is expanded in powers of \tilde{U}_k ,

$$\det[\delta_{ij} + \tilde{V}_k^{ij}] \approx C + F_1(\tilde{U}_k) + F_2(\tilde{U}_k^2) + \dots, \quad (18)$$

where C contains field-independent terms, $F_1(\tilde{U}_k)$ and $F_2(\tilde{U}_k^2)$ represent the linear and quadratic terms in the periodic part of the potential. Deriving the WH-RG equation (17) with respect to one of the field variables and inserting the expanded form of the potential into it, the WH-RG equation (17) becomes

$$(2 + k\partial_k)\frac{d}{d\varphi_1}\tilde{U}_k(\varphi_1, \dots, \varphi_N) = -\frac{1}{4\pi} \frac{\frac{d}{d\varphi_1}[F_1(\tilde{U}_k) + \mathcal{O}(\tilde{U}_k^2)]}{C + F_1(\tilde{U}_k) + \mathcal{O}(\tilde{U}_k^2)}. \quad (19)$$

Since the constant term C is field independent, equation (19) can be rewritten as

$$(2 + k\partial_k)\frac{d}{d\varphi_1}\tilde{U}_k(\varphi_1, \dots, \varphi_N) = -\frac{1}{4\pi} \frac{\frac{d}{d\varphi_1}\left[\frac{F_1(\tilde{U}_k) + \mathcal{O}(\tilde{U}_k^2)}{C}\right]}{1 + \frac{F_1(\tilde{U}_k) + \mathcal{O}(\tilde{U}_k^2)}{C}}. \quad (20)$$

The mass-corrected UV-RG equation can be achieved by linearizing equation (20) and reads

$$(2 + k\partial_k)\tilde{U}_k(\varphi_1, \dots, \varphi_N) \approx -\frac{1}{4\pi} \frac{F_1(\tilde{U}_k)}{C}. \quad (21)$$

4. Flavour-doublet layered sine-Gordon model

4.1. Definition and rotation

In this section we discuss the rotation of the LSG model with $N = 2$ layers and apply the mass-corrected UV WH-RG method in order to map out the phase structure of the model. The ansatz for the blocked potential should preserve all symmetries of the original model at the UV cutoff scale $k = \Lambda$ and should be rich enough to contain all the interactions which are generated during the RG flow. Therefore, the specialization of equation (1) to the case of two layers yields

$$\begin{aligned} \mathcal{L}_{2\text{LSG}} = & \frac{1}{2} \sum_{i=1}^2 (\partial\varphi_i)^2 + \frac{1}{2} J(\varphi_1 - \varphi_2)^2 \\ & + \sum_{n,m=0}^{\infty} [u_{nm} \cos(n\beta\varphi_1) \cos(m\beta\varphi_2) + v_{nm} \sin(n\beta\varphi_1) \sin(m\beta\varphi_2)], \end{aligned} \quad (22)$$

where the Fourier decomposition of the periodic part has a general form. All couplings u_{nm} and v_{nm} are dimensionful. For $N = 2$ layers the mass eigenvalues are $0, 2J$. The particular choice of $\beta = 2\sqrt{\pi}$ for the LSG represents the bosonized version of the two-flavour massive Schwinger model.

In order to emphasize the symmetries of the LSG model we now apply a rotation of the field variables described in [15],

$$\varphi_1 \rightarrow \frac{\alpha_1 + \alpha_2}{\sqrt{2}}, \quad \varphi_2 \rightarrow \frac{\alpha_1 - \alpha_2}{\sqrt{2}}, \quad (23)$$

where the periodic part of the blocked potential

$$U_k(\varphi_1, \varphi_2) = \sum_{n,m=0}^{\infty} [u_{nm} \cos(n\beta\varphi_1) \cos(m\beta\varphi_2) + v_{nm} \sin(n\beta\varphi_1) \sin(m\beta\varphi_2)] \quad (24)$$

has the following rotated form

$$\begin{aligned} U_k(\alpha_1, \alpha_2) = & \sum_{n,m=0}^{\infty} \frac{u_{nm} + v_{nm}}{2} \cos\left[(n-m)\frac{\beta}{\sqrt{2}}\alpha_1\right] \cos\left[(n+m)\frac{\beta}{\sqrt{2}}\alpha_2\right] \\ & + \sum_{n,m=0}^{\infty} \frac{u_{nm} - v_{nm}}{2} \cos\left[(n+m)\frac{\beta}{\sqrt{2}}\alpha_1\right] \cos\left[(n-m)\frac{\beta}{\sqrt{2}}\alpha_2\right] \\ & - \sum_{n,m=0}^{\infty} \frac{u_{nm} + v_{nm}}{2} \sin\left[(n-m)\frac{\beta}{\sqrt{2}}\alpha_1\right] \sin\left[(n+m)\frac{\beta}{\sqrt{2}}\alpha_2\right] \\ & + \sum_{n,m=0}^{\infty} \frac{v_{nm} - u_{nm}}{2} \sin\left[(n+m)\frac{\beta}{\sqrt{2}}\alpha_1\right] \sin\left[(n-m)\frac{\beta}{\sqrt{2}}\alpha_2\right]. \end{aligned} \quad (25)$$

The general form of the rotated periodic potential reads

$$U_k = \sum_{n,m=0}^{\infty} [f_{nm} \cos(nb\alpha_1) \cos(mb\alpha_2) + h_{nm} \sin(nb\alpha_1) \sin(mb\alpha_2)], \quad (26)$$

where the rotated frequency $b = \beta/\sqrt{2}$. Some identifications read $f_{02} = \frac{1}{2}(u_{11} + v_{11})$, $f_{20} = \frac{1}{2}(u_{11} - v_{11})$ and $f_{11} = u_{01} + u_{10}$. Finally, the rotated Lagrangian reads

$$\mathcal{L}_{\text{LSG}} = \frac{1}{2}(\partial\alpha_1)^2 + \frac{1}{2}(\partial\alpha_2)^2 + \frac{1}{2}M_2^2\alpha_2^2 + U_k(\alpha_1, \alpha_2). \quad (27)$$

Note that the field α_1 has no explicit mass term but for α_2 the explicit mass $M_2^2 = J/2$ breaks the periodicity. Therefore, the model is disentangled into a ‘periodic’ mode α_1 and a non-periodic field α_2 .

4.2. Wegner–Houghton RG approach to the rotated flavour-doublet model

The specialization of the dimensionless WH–RG equation (15) for two layers can be written as

$$(2 + k\partial_k)\tilde{V}_k(\alpha_1, \alpha_2) = -\frac{k^2}{4\pi} \ln\left([1 + \tilde{V}_k^{11}][1 + \tilde{V}_k^{22}] - [\tilde{V}_k^{12}]^2\right), \quad (28)$$

where $\tilde{V}_k^{ij} = \partial_{\alpha_i}\partial_{\alpha_j}\tilde{V}_k(\alpha_1, \alpha_2)$. Starting with a general form for the dimensionless rotated blocked potential for the flavour-doublet LSG model

$$\tilde{V}_k = \frac{1}{2}\tilde{M}_2^2\alpha_2^2 + \sum_{n,m=0}^{\infty} [\tilde{f}_{nm} \cos(nb\alpha_1) \cos(mb\alpha_2) + \tilde{h}_{nm} \sin(nb\alpha_1) \sin(mb\alpha_2)], \quad (29)$$

where $\tilde{f}_{nm} = k^{-2}f_{nm}$ and $\tilde{h}_{nm} = k^{-2}h_{nm}$ are the dimensionless coupling constants and using the ‘mass-corrected’ UV approximation of the WH–RG equation (28), which is based on equation (21), this reduces to a set of uncoupled differential equations for the coupling parameters of the model

$$\begin{aligned} (2 + k\partial_k)\tilde{f}_{nm}(k) &= \frac{1}{4\pi} \frac{k^2 m^2 b^2 + (k^2 + M_2^2)n^2 b^2}{(k^2 + M_2^2)} \tilde{f}_{nm}, \\ (2 + k\partial_k)\tilde{h}_{nm}(k) &= \frac{1}{4\pi} \frac{k^2 m^2 b^2 + (k^2 + M_2^2)n^2 b^2}{(k^2 + M_2^2)} \tilde{h}_{nm}, \end{aligned} \quad (30)$$

where the dimensionful mass M_2^2 is scale independent. The UV approximated RG flow equations are decoupled, and their solution can be obtained analytically:

$$\begin{aligned} \tilde{f}_{nm}(k) &= \tilde{f}_{nm}(\Lambda) \left(\frac{k^2 + M_2^2}{\Lambda^2 + M_2^2} \right)^{\frac{m^2 b^2}{8\pi}} \left(\frac{k}{\Lambda} \right)^{-2 + \frac{b^2 n^2}{4\pi}}, \\ \tilde{h}_{nm}(k) &= \tilde{h}_{nm}(\Lambda) \left(\frac{k^2 + M_2^2}{\Lambda^2 + M_2^2} \right)^{\frac{m^2 b^2}{8\pi}} \left(\frac{k}{\Lambda} \right)^{-2 + \frac{b^2 n^2}{4\pi}}. \end{aligned} \quad (31)$$

Here $\tilde{f}_{nm}(\Lambda)$ and $\tilde{h}_{nm}(\Lambda)$ are the initial conditions at the UV cutoff $k = \Lambda$. Using the solution (31), one can read off the IR scaling of the various Fourier amplitudes. In the IR limit ($k \rightarrow 0$) all the purely non-periodic modes ($n = 0$) become relevant, i.e. $\tilde{f}_{0m} \propto k^{-2}$. The periodic modes \tilde{f}_{nm} and \tilde{h}_{nm} (with $n > 0$) may be relevant or irrelevant, depending on the value of b^2 . If $b^2 > 8\pi$, the RG flow of all the periodic modes tends to zero, and if $b^2 < 8\pi$, there is at least one mode which becomes relevant in the IR limit. We recall that $b^2 = \beta^2/2$. Therefore, the critical value which separates the two scaling regime of the original LSG model is $\beta_c^2 = 16\pi$. We would like to recall that the LSG model with $N = 1$ layer is the 2D-SG model with $\beta_c^2 = 8\pi$. So, in the case of $N = 2$ layers the critical value is increased compared to that of the 2D-SG model.

5. Flavour-triplet layered sine-Gordon model

5.1. Definition and rotation

The Lagrangian of the LSG model with $N = 3$ layers can be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \sum_{i=1}^3 (\partial\varphi_i)^2 + \frac{1}{2} J \sum_{i=1}^2 (\varphi_{i+1} - \varphi_i)^2 \\ &+ \sum_{n,m,l=-\infty}^{\infty} w_{nml} \exp(in\beta\varphi_1) \exp(im\beta\varphi_2) \exp(il\beta\varphi_3). \end{aligned} \quad (32)$$

The parameters w_{nml} of the Fourier decomposition are dimensionful quantities. The mass matrix with eigenvalues $(0, J, 3J)$ reads explicitly

$$\underline{\underline{M}}^2 = \begin{pmatrix} J & -J & 0 \\ -J & 2J & -J \\ 0 & -J & J \end{pmatrix}. \quad (33)$$

The explicit form of the rotation of the field variables in the case of three layers has the following structure:

$$\varphi_1 \rightarrow \frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{2}} + \frac{\alpha_3}{\sqrt{6}}, \quad \varphi_2 \rightarrow \frac{\alpha_1}{\sqrt{3}} - \frac{\sqrt{2}\alpha_3}{\sqrt{3}}, \quad \varphi_3 \rightarrow \frac{\alpha_1}{\sqrt{3}} + \frac{\alpha_2}{\sqrt{2}} + \frac{\alpha_3}{\sqrt{6}}. \quad (34)$$

For illustrative purposes the transformation of the periodic part of the bare potential is discussed by taking into account only the fundamental modes. In this case the bare potential has a flavour symmetry ($\varphi_1 \longleftrightarrow \varphi_3$) and reads

$$U(\varphi_1, \varphi_2, \varphi_3) = u \cos(\beta\varphi_1) + u_2 \cos(\beta\varphi_2) + u \cos(\beta\varphi_3), \quad (35)$$

where $w_{100} = w_{001} \equiv u/2$ and $w_{010} \equiv u_2/2$ are introduced. Applying the rotation (34) on the periodic potential (35) the transformed potential is

$$\begin{aligned} U(\alpha_1, \alpha_2, \alpha_3) &= u_2 \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{2\beta}{\sqrt{6}}\alpha_3\right) \\ &+ 2u \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \cos\left(\frac{\beta}{\sqrt{6}}\alpha_3\right) \\ &+ 2u \sin\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \sin\left(\frac{\beta}{\sqrt{6}}\alpha_3\right) \\ &+ 2u_2 \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \sin\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \sin\left(\frac{\beta}{\sqrt{6}}\alpha_3\right). \end{aligned} \quad (36)$$

In general, the rotated form of the blocked periodic potential reads

$$U_k = \sum_{n,m,l=-\infty}^{\infty} j_{nml} \exp\left(\frac{in\beta}{\sqrt{3}}\alpha_1\right) \exp\left(\frac{im\beta}{\sqrt{2}}\alpha_2\right) \exp\left(\frac{il\beta}{\sqrt{6}}\alpha_3\right), \quad (37)$$

where j_{nml} are the transformed expansion coefficients. The new frequency for the periodic mode α_1 is $b_1 = \beta/\sqrt{3}$. For the two non-periodic modes (with explicit mass terms), the transformed frequencies read $b_2 = \beta/\sqrt{2}$ and $b_3 = \beta/\sqrt{6}$. After rotation the Lagrangian of the ($N = 3$)-layer model is

$$\mathcal{L}_{3\text{LSG}} = \sum_{i=1}^3 \frac{1}{2} (\partial\alpha_i)^2 + \frac{1}{2} M_2^2 \alpha_2^2 + \frac{1}{2} M_3^2 \alpha_3^2 + U_k(\alpha_1, \alpha_2, \alpha_3), \quad (38)$$

with mass eigenvalues $M_2^2 = J$ and $M_3^2 = 3J$. As in the two-layer case we have decomposed the three-layer model into one periodic mode α_1 and two non-periodic fields α_2, α_3 .

5.2. Wegner–Houghton RG approach to the rotated flavour-triplet model

We generalize the treatment discussed in section 4.2 to the case of three layers by repeating the same steps as in section 4.2. The rotated dimensionless blocked potential for the flavour-triplet LSG model is

$$\begin{aligned} \tilde{V}_k(\alpha_1, \alpha_2, \alpha_3) &= \frac{1}{2} \tilde{M}_2^2 \alpha_2^2 + \frac{1}{2} \tilde{M}_3^2 \alpha_3^2 \\ &+ \sum_{n,m,l=-\infty}^{\infty} \tilde{j}_{nml} \exp(inb_1\alpha_1) \exp(imb_2\alpha_2) \exp(ilb_3\alpha_3). \end{aligned} \quad (39)$$

The dimensionless WH–RG equation in $d = 2$ dimensions, for three fields $\alpha_{1,2,3}$, reads

$$\begin{aligned} (2 + k\partial_k) \tilde{V}_k &= -\frac{k^2}{4\pi} \ln \left([1 + \tilde{V}_k^{11}] [1 + \tilde{V}_k^{22}] [1 + \tilde{V}_k^{33}] - [1 + \tilde{V}_k^{22}] [\tilde{V}_k^{13}]^2 - [1 + \tilde{V}_k^{33}] [\tilde{V}_k^{12}]^2 \right. \\ &\quad \left. - [1 + \tilde{V}_k^{11}] [\tilde{V}_k^{23}]^2 + [\tilde{V}_k^{12}] [\tilde{V}_k^{23}] [\tilde{V}_k^{31}] + [\tilde{V}_k^{13}] [\tilde{V}_k^{21}] [\tilde{V}_k^{32}] \right). \end{aligned} \quad (40)$$

where $\tilde{V}_k^{ij} = \partial_{\alpha_i} \partial_{\alpha_j} \tilde{V}_k(\alpha_1, \alpha_2, \alpha_3)$ is the second derivative of the dimensionless blocked potential with respect to the field variables. In order to obtain reliable UV scaling laws which

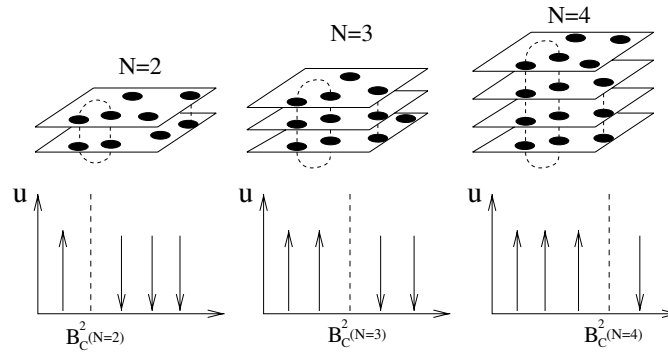


Figure 1. We illustrate the schematic RG trajectories of the multi-layer sine-Gordon model with $N = 2, 3, 4$ layers in the plane ($B^2 \equiv \beta^2$, $u \equiv \tilde{u}_{01}$) and the shift of the critical value $B_c^2(N) \equiv \beta_c^2(N) = 8N\pi$. Each layer corresponds to a sine-Gordon model which are coupled by the coupling J . The solid discs represent the topological excitation of the layered system.

can be used to determine the phase diagram of the LSG model with $N = 3$ layers one has to use the mass-corrected UV approximation for the WH–RG equation (40). This reduces to a set of uncoupled differential equations for the coupling constants of the model

$$(2 + k\partial_k)\tilde{J}_{nml}(k) = \frac{1}{4\pi} \left(n^2 b_1^2 + \frac{k^2 m^2 b_2^2}{k^2 + M_2^2} + \frac{k^2 l^2 b_3^2}{k^2 + M_3^2} \right) \tilde{J}_{nml}. \quad (41)$$

The solution can be obtained analytically,

$$\tilde{J}_{nml}(k) = \tilde{J}_{nml}(\Lambda) \left(\frac{k}{\Lambda} \right)^{-2 + \frac{b_1^2 n^2}{4\pi}} \left(\frac{k^2 + M_2^2}{\Lambda^2 + M_2^2} \right)^{\frac{m^2 b_2^2}{8\pi}} \left(\frac{k^2 + M_3^2}{\Lambda^2 + M_3^2} \right)^{\frac{l^2 b_3^2}{8\pi}}, \quad (42)$$

where $\tilde{J}_{nml}(\Lambda)$ is the initial condition at the UV cutoff $k = \Lambda$. In the IR regime ($k \rightarrow 0$), the pure non-periodic modes are relevant (increasing) coupling constants $\tilde{J}_{0ml} \propto k^{-2}$ independently of b_i^2 . The periodic modes \tilde{J}_{nml} , $n > 0$ are found to be relevant or irrelevant couplings depending on the value of b_1^2 . If $b_1^2 > 8\pi$, the RG flow of all the periodic modes tends to zero, and if $b_1^2 < 8\pi$, it becomes relevant in the IR limit. We recall that $b_1^2 = \beta^2/3$. Therefore, the critical value for the original three-layer LSG model is $\beta_c^2 = 24\pi$. This suggest that the critical value of the parameter which separates the two phases of LSG model with N layers depends on the number of layers, $\beta_c^2 = N8\pi$ (see figure 1). This is in agreement with the perturbative results obtained in [15].

6. 3D-SG model

Since in the continuum limit $N \rightarrow \infty$ the N -layer LSG model can be considered as the discretized version of the 3D-SG model, one can clarify the previously obtained layer dependence of the critical parameter $\beta_c^2 = N8\pi$ by investigating the phase structure of the 3D-SG model. One might expect that for $N \rightarrow \infty$ the phase structure of the LSG model recovers that of the 3D-SG model. The action for the 3D-SG model reads

$$S = \int d^3r \left[\frac{1}{2} (\partial_\mu \varphi)^2 + u_{3D} \cos(\beta_{3D} \varphi) \right], \quad (43)$$

where β_{3D} , u_{3D} are the dimensionful parameters of the theory. The corresponding dimensionless quantities are $\tilde{\beta}^2 = k\beta_{3D}^2$ and $\tilde{u} = k^{-3}u_{3D}$. The WH–RG approach to

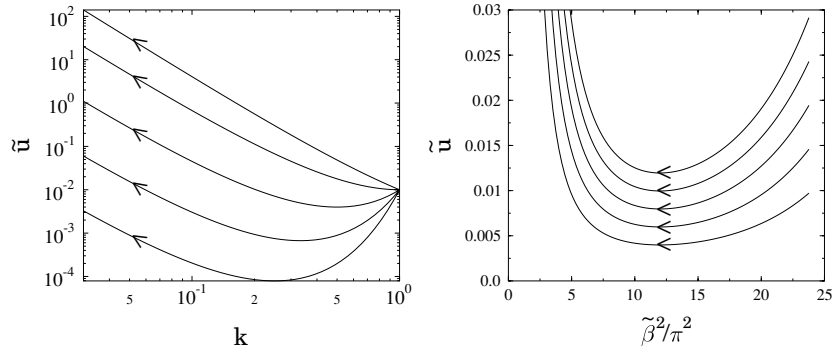


Figure 2. On the left panel, the scaling of the dimensionless coupling constant $\tilde{u}(k)$ of the 3D-SG model is represented graphically for various initial values for the dimensionless parameter $\tilde{\beta}^2(\Lambda = 1) = 4\pi^2, 12\pi^2, 24\pi^2, 36\pi^2, 48\pi^2$ (from top to bottom). In the IR limit ($k \rightarrow 0$) the coupling constant \tilde{u} always becomes a relevant (increasing) parameter independently of $\tilde{\beta}^2(\Lambda)$ which demonstrates that the 3D-SG model has a single phase in the local potential approximation. On the right panel, the RG flow diagram of the 3D-SG model also illustrates the existence of the single phase of the model and the infinite value of β_c^2 . The arrows indicate the direction of the flow.

the 3D-SG model has been developed and discussed in [20]. Using the local potential approximation, the dimensionless WH–RG equation reads

$$\left(3 - \frac{1}{2}\tilde{\varphi}\partial_{\tilde{\varphi}} + k\partial_k\right)\tilde{V}_k(\tilde{\varphi}) = -\frac{1}{4\pi^2}\ln\left(1 + \partial_{\tilde{\varphi}}^2\tilde{V}_k(\tilde{\varphi})\right). \quad (44)$$

The UV approximation for equation (44) can be achieved by the linearization of the logarithm around the Gaussian fixed point and results in

$$(3 + k\partial_k)\tilde{u}(k) = \frac{1}{4\pi^2}\tilde{\beta}^2(k)\tilde{u}(k), \quad k\partial_k\tilde{\beta}^2(k) = \tilde{\beta}^2(k). \quad (45)$$

with the solution

$$\begin{aligned} \tilde{u}(k) &= \tilde{u}(\Lambda)\left(\frac{k}{\Lambda}\right)^{-3}\exp\left\{\frac{\tilde{\beta}^2(\Lambda)}{4\pi^2}\left[\left(\frac{k}{\Lambda}\right) - 1\right]\right\} \\ \tilde{\beta}^2(k) &= \tilde{\beta}^2(\Lambda)\left(\frac{k}{\Lambda}\right) \end{aligned} \quad (46)$$

where $\tilde{\beta}(\Lambda)$ and $\tilde{u}(\Lambda)$ are the bare values of the couplings. In the IR limit, if $k \rightarrow 0$ the coupling constant $\tilde{u}(k)$ always becomes a relevant parameter ($\tilde{u} \rightarrow \infty$) independently of $\tilde{\beta}^2$ (see figure 2). Therefore, the 3D-SG model has only a single phase within the LPA.

On the one hand, in the case of the LSG model for the bulk limit ($N \rightarrow \infty$), the critical value which separates the two phases of the layered model becomes infinitely large ($\beta_c^2 \rightarrow \infty$) and the model has only a single phase. On the other hand, in this continuum limit, the multi-layer model can be considered as the discretized version of the 3D-SG model which has been shown to have a single phase, within in the local potential approximation (see figure 2 and [20]). We conclude that the latter observation is entirely consistent with the infinite value of β_c^2 in the continuum limit.

7. Summary

The phase structure of the layered sine-Gordon (LSG) model with N layers has been analysed in terms of a non-perturbative renormalization group (RG) treatment with a sharp momentum

cutoff. The LSG model consists of N coupled SG models each of which corresponds to a specific layer. The coupling between the layers is described by a quadratic term which can be considered as a mass term. All the Lagrangians studied in the paper have the general structure (2). The case with a mass matrix that has exactly one non-vanishing mass eigenvalue has been discussed in detail.

The LSG model has relevance both in high-energy and, perhaps even more importantly, in low-temperature physics. The N -layer SG model is the bosonized version of the N -flavour Schwinger model, and the double-layer SG model has been used to describe the vortex properties of high transition temperature superconductors [11, 19].

Previously, models of this type, with up to two layers, were analysed in terms of the Wegner–Houghton renormalization group (WH–RG) method [13, 20]. We have described here the generalization of the RG analysis for N layers, which has allowed us to consider the dependence of the phase structure on the number of the layers. In order to be able to compare the results of our RG analysis to that of the perturbative treatment performed for the LSG model [15], we have performed a rotation of the fields before applying the WH–RG method.

It has been demonstrated that the LSG model undergoes a Kosterlitz–Thouless-type phase transition where the critical value which separates the two phases of the model depends on the number of layers, $\beta_c^2 = N8\pi$. Therefore, the transition ‘temperature’ in the layered case was found to differ from a one-layer ‘pure’ sine-Gordon model. Furthermore, we have shown (see also [15]) that this conjecture finds a natural explanation after a suitable linear transformation of the field variables, which corresponds to a rotation in the internal space towards a frame in which the mass matrix is diagonal.

The LSG model in the continuum limit $N \rightarrow \infty$ has been shown to be considered as the discretized version of the three-dimensional SG model which has a single phase within the local potential approximation. The infinite critical value of the parameter β_c^2 for the LSG model in the continuum limit is entirely consistent with the latter observation.

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References

- [1] Kosterlitz J M 1974 *J. Phys. C: Solid State Phys.* **7** 1046
- [2] Amit D, Goldschmidt Y Y and Grinstein G 1980 *J. Phys. A: Math. Gen.* **13** 585
- [3] Balogh J and Hegedűs A 2000 *J. Phys. A: Math. Gen.* **33** 6543
Faber M and Ivanov A N 2003 *J. Phys. A: Math. Gen.* **36** 7839 (Preprint [hep-th/0306229](#))
Bozkaya H *et al* 2005 *Preprints* [hep-th/0505276](#) and [hep-th/0512286](#)
- [4] Coleman S 1975 *Phys. Rev. D* **11** 2088
- [5] Nándori I, Polonyi J and Sailer K 2001 *Phys. Rev. D* **63** 045022
Nándori I, Polonyi J and Sailer K 2001 *Phil. Mag. B* **81** 1615
- [6] Nagy S, Polonyi J and Sailer K 2004 *Phys. Rev. D* **70** 105023
Nagy S, Polonyi J and Sailer K 2004 *Acta Phys. Hung. A* **19** 247
Nagy S and Sailer K 2001 *Phil. Mag. B* **81** 1583–6
- [7] Fischler W, Kogut J and Susskind L 1979 *Phys. Rev. D* **19** 1188
- [8] Hetrick J E, Hosotani Y and Iso S 1995 *Phys. Lett. B* **350** 92
- [9] Banks T, Horn D and Neuberger H 1976 *Nucl. Phys. B* **108** 119
- [10] Halpern M B 1975 *Phys. Rev. D* **12** 1684

- [11] Pierson S W and Valls O T 1992 *Phys. Rev. B* **45** 13076
Pierson S W 1995 *Phys. Rev. Lett.* **74** 2359
Pierson S W 1997 *Phys. Rev. B* **55** 14536
- [12] Vad K, Mészáros S and Sas B 2005 *Physica C* **432** 43–52
Vad K *et al* 2006 *Phil. Mag.* **86** 2115
Pethes I *et al* 2003 *Phys. Rev. B* **68** 184509
Sas B *et al* 2000 *Phys. Rev. B* **61** 9118
- [13] Nándori I, Nagy S, Sailer K and Jentschura U D 2005 *Nucl. Phys. B* **725** 467
- [14] Wegner F J and Houghton A 1973 *Phys. Rev. A* **8** 401
- [15] Jentschura U D, Nándori I and Zinn-Justin J 2005 *Preprint* [hep-th/0509186](#) (*Ann. Phys.* at press)
- [16] Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
Appelquist T and Carrazone J 1975 *Phys. Rev. D* **11** 2856–61
- [17] Wilson K G 1971 *Phys. Rev. D* **3** 1818
- [18] Polonyi J 2004 *Central Eur. J. Phys.* **1** 1
Pawlowski J 2005 *Preprint* [hep-th/0512261](#)
- [19] Nándori I and Sailer K 2005 *Phil. Mag.* **86** 2033
- [20] Nándori I, Sailer K, Jentschura U D and Soff G 2004 *Phys. Rev. D* **69** 025004
Nándori I, Sailer K, Jentschura U D and Soff G 2002 *J. Phys. G: Nucl. Part. Phys.* **28** 607